AV-8874 ①

B.Sc. (Hon's) (Third Semester) Exam. 2015-16

Mathematics (Differential Equations)

Model Answer / Suggestive Answer

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1 (i). order - 2, degree - 1.

(ii). $\cancel{\frac{dy}{dx} + P\frac{dy}{dx} = 0}$ $\frac{dy}{dx} + Py = Q$ or $\frac{dx}{dy} + Px = Q$.

(iii). $I.F. = e^{\int \cot x\, dx}$
$= \sin x$.

(iv). $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

$\Rightarrow (D^2 - 3D + 2)y = 0$

A.E. $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0$

$m = 1, 2$

$\therefore y = c.f. = c_1 e^x + c_2 e^{2x}$.

(v). $cf = e^{\alpha x}\left[c_1 \cosh\sqrt{\beta}x + c_2 \sinh\sqrt{\beta}x\right]$.

(vi). $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$.

(vii). $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$

(viii). $cf = e^x$.

2(a). Given $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Put $x = X+h$, $y = Y+K$.

$$\frac{dY}{dX} = \frac{X+2Y+(h+2K-3)}{2X+Y+(2h+K-3)}$$

for h and K, Put $\left.\begin{array}{l} h+2K-3=0 \\ 2h+K-3=0 \end{array}\right\} h=1, K=1$

$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$ (homo.) $\longrightarrow$ (I)

Put $Y = vX \Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX} \longrightarrow$ (II)

$$x\frac{dv}{dx}=\frac{1-v^2}{2+v} \Rightarrow \int\frac{dx}{x}=\int\frac{2+v}{1-v^2}dv+c$$

$$\log x=\frac{1}{2}\log(1+v)-\frac{3}{2}\log(1-v)+\log c$$

$$\Rightarrow x^2=\frac{(1+v)}{(1-v)^3}\cdot c^2$$

$$\Rightarrow x^2=\frac{1+y/x}{(-y/x)^3}\cdot c^2$$

$$(x+y)\cdot c^2=(x-y)^3 \Rightarrow (x+y-2)\cdot c_1^*=(x-y)^3.$$

(b). Given $y-x\frac{dy}{dx}=a\left(y^2+\frac{dy}{dx}\right)$

$$\Rightarrow y-ay^2=(x+a)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y(1-ay)}{x+a} \Rightarrow \int\frac{dx}{x+a}=\int\frac{dy}{y(1-ay)}+\log c$$

$$\log(x+a)=\int\left(\frac{1}{y}+\frac{a}{1-ay}\right)dy+\log c$$

$$=\log y+\cancel{a}\cdot\frac{\log(1-ay)}{\cancel{-a}}+\log c$$

$$(x+a)=c\cdot y/(1-ay)$$

3(a). Given $(1-x^2)\frac{dy}{dx}+2xy=x(1-x^2)^{1/2}$

$$\Rightarrow \frac{dy}{dx}+\frac{2x}{1-x^2}y=\frac{x(1-x^2)^{1/2}}{1-x^2}$$

$$P=\frac{2x}{1-x^2},\ Q=\frac{x}{\sqrt{1-x^2}}$$

$$I.F=e^{\int\frac{2x}{1-x^2}dx}=e^{-\log(1-x^2)}=1/1-x^2$$

Sol. $y\cdot\frac{1}{1-x^2}=\int\frac{x}{\sqrt{1-x^2}}\cdot\frac{1}{1-x^2}dx+c=\frac{1}{\sqrt{1-x^2}}+c$

or $y=\sqrt{1-x^2}+c(1-x^2)$

P.T.O.

3(b). Given $\frac{dy}{dx} = x^3y^3 - xy$

$\Rightarrow \frac{dy}{dx} + xy = x^3y^3$

$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3$

Put $1/y^2 = v \Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow -\frac{1}{2}\frac{dv}{dx} + xv = x^3$

$\Rightarrow \frac{dv}{dx} - 2xv = -2x^3$

$P = -2x, \; Q = -2x^3$

I.F. $= e^{\int -2x\,dx} = e^{-x^2}$

Sol. $v \cdot e^{-x^2} = \int e^{-x^2} \cdot (-2x^3)\,dx + C$

$\frac{e^{-x^2}}{y^2} = -2\int e^{-x^2} \cdot x^2 \cdot x\,dx + C$

Put $x^2 = t$

$2x\,dx = dt$

$= -\int e^{-t} \cdot t\,dt + C$

$= -[\cancel{-}t \cdot e^{-t} - e^{-t}] + C$

$\frac{e^{-x^2}}{y^2} = + e^{-x^2}(x^2+1) + C.$

4(a) Given $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = e^{2x}(1+x)$

$\Rightarrow (D^3 - 7D - 6)y = e^{2x}(1+x)$

A.E. $m^3 - 7m - 6 = 0$

$m^3 + m^2 - m^2 - m - 6m - 6 = 0$

P.T.O

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$m^2(m+1) - m(m+1) - 6(m+1) = 0$

$\Rightarrow (m+1)(m^2 - m - 6) = 0$

$(m+1)(m+2)(m-3) = 0$

$cf = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

$P.I. = \frac{1}{D^3 - 7D - 6} e^{2x}(1+x)$

$= e^{2x} \frac{1}{(D+2)^3 - 7(D+2) - 6} (1+x)$

$= e^{2x} \frac{1}{D^3 + 6D^2 + 5D - 12} \cdot (1+x)$

$= \frac{e^{2x}}{-12} \left[1 - \frac{1}{12}(D^3 + 6D^2 + 5D)\right]^{-1} \cdot (1+x)$

$= \frac{e^{2x}}{-12} \left(1 + \frac{5}{12} D\right)(1+x)$

$= \frac{e^{2x}}{-12}\left(1 + x + \frac{5}{12}\right) = -\frac{1}{144} e^{2x}(12x+17)$

$y = cf + P.I. = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{e^{2x}}{144}(12x+17)$

4(b) $(D^3 + 3D^2 + 3D + 1)y = e^{-x}$

A.E. $m^3 + 3m^2 + 3m + 1 = 0$

$(m+1)^3 = 0 \Rightarrow m = -1, -1, -1$

$cf = (c_1 + c_2 x + c_3 x^2) e^{-x}$

$P.I. = \frac{1}{(D+1)^3} e^{-x} = \frac{1}{(-1+1)^3} e^{-x}$

$\because f(-1) = 0$

$\therefore P.I. = x \cdot \frac{1}{3(D+1)^2} e^{-x}$

$= \frac{x^3}{6} e^{-x}$

$\therefore y = (c_1 + c_2 x + c_3 x^2) e^{-x} + \frac{x^3}{6} e^{-x}$

P.T.O.

5. Given $x^2 \frac{d^3y}{dx^3} + 3x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 \log x$

$\Rightarrow x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = x^3 \log x$

Put $x = e^z \Rightarrow z = \log x$

$\Rightarrow \{D(D-1)(D-2) + 3D(D-1) + D\}y = e^{3z} \cdot z$

$\Rightarrow (D^3 - \cancel{3D^2} + \cancel{2D} + \cancel{3D^2} - \cancel{3D} + \cancel{D})y = e^{3z} \cdot z$

$\Rightarrow D^3 y = e^{3z} \cdot z$

A.E. $m^3 = 0 \Rightarrow m = 0, 0, 0$

$CF = c_1 + c_2 z + c_3 z^2 = c_1 + c_2 \log x + c_3 (\log x)^2$

$P.I. = \frac{1}{D^3} e^{3z} \cdot z = e^{3z} \cdot \frac{1}{(D+3)^3} \cdot z$

$= \frac{e^{3z}}{27}(z-1) = \frac{x^3}{27}(\log x - 1)$

$y = c_1 + c_2 \log x + c_3 (\log x)^2 + \frac{x^3}{27}(\log x - 1)$.

6. Given

$$3x \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \longrightarrow (1)$$

Let $y = \sum_{n=0}^{\infty} a_n x^{m+n}$ be a sol. of (1)

Then $\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1}$

$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}$

Putting above value in (1), we get

$$3 \sum a_n (m+n)(m+n-1) x^{m+n-1} + 2 \cdot \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1} + \sum a_n x^{m+n} = 0$$

P.T.O.

(6)

$$\sum_{n=0}^{\infty} a_n(m+n)[3(m+n-1)+2]x^{m+n-1} + \sum a_n x^{m+n} = 0$$

$$\sum_{n=0}^{\infty} a_n(m+n)(3m+3n-1)x^{m+n-1} + \sum a_n x^{m+n} = 0$$

Equating the coeff. of x^{m-1}

$$m(3m-1) = 0 \Rightarrow m = 0 \text{ and } m = 1/3$$

Again equating the coeff. of x^{m+n-1}

$$a_n(m+n)(3m+3n-1) + a_{n-1} = 0$$

$$\Rightarrow a_n = \frac{-a_{n-1}}{(m+n)(3m+3n-1)}$$

Case I When $m=0$

$$a_n = -\frac{a_{n-1}}{n(3n-1)} \quad \text{Put } n = 1, 2, 3, 4, \ldots$$

$$a_1 = -\frac{a_0}{2\cdot 1} = -\frac{a_0}{2!}$$

$$a_2 = -\frac{a_1}{2\cdot 5} = \frac{a_0}{20}$$

$$a_3 = -\frac{a_2}{3\cdot 8} = \frac{a_0}{24\cdot 20}$$

$\vdots$

$$\therefore y = a_0\left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} \cdots\right).$$

Similarly for $m = 1/3$ Also find.

$$y = a_0\left(1 - \frac{x}{4} + \frac{x^2}{56} \cdots\right).$$

7. Given

$$(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$$

P.T.O.

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{2x+5}{x+2}\frac{dy}{dx} + \frac{2y}{x+2} = \left(\frac{x+1}{x+2}\right)e^{x} \longrightarrow ①$$

$$P = -\left(\frac{2x+5}{x+2}\right),\ Q = \frac{2}{x+2},\ R = \left(\frac{x+1}{x+2}\right)e^{x}.$$

$$1 + \frac{P}{2} + \frac{Q}{2^2} = 0$$

$\therefore y = e^{2x}$ is a part of c.f.

Put $y = e^{2x}\cdot v$ in ①

$$\frac{d^2v}{dx^2} + \frac{2x+3}{x+2}\frac{dv}{dx} = \frac{x+1}{x+2}e^{-x}$$

$$\Rightarrow \frac{dz}{dx} + \frac{2x+3}{x+2}z = \frac{x+1}{x+2}e^{-x} \qquad \left(\because \frac{dv}{dx} = z\right)$$

which is linear in z.

$$\text{I.F.} = e^{\int \frac{2x+3}{x+2}} = \frac{e^{2x}}{x+2}$$

$$\text{Sol. } z\cdot\frac{e^{2x}}{x+2} = \int \frac{e^{2x}}{x+2}\cdot\frac{x+1}{x+2}e^{-x}dx + c_1$$

$$= \int \frac{e^{x}(x+1)}{(x+2)^2}dx + c_1$$

$$\frac{dv}{dx} = z = e^{-x} + c_1(x+2)e^{-2x}$$

$$v = \int\left[e^{-x} + c_1(x+2)e^{-2x}\right]dx + c_2$$

$$= -e^{-x} + c_1\frac{e^{-2x}}{4}(2x+5) + c_2$$

$$\therefore y = u\cdot v = e^{2x}\left[-e^{-x} + c_1\frac{e^{-2x}}{4}(2x+5) + c_2\right]$$

P.T.O.

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8. we know that

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$\Rightarrow J_n' = \sum \frac{(-1)^r}{r!\,\Gamma(n+r+1)} (n+2r)\left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

$$\Rightarrow xJ_n' = \sum \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum \frac{(-1)^r (2n+2r-n)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$xJ_n' = xJ_{n-1} - nJ_n$$ proved

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